

A GRÜSS INEQUALITY FOR n -POSITIVE LINEAR MAPS

MOHAMMAD SAL MOSLEHIAN¹ AND RAJNA RAJIĆ²

ABSTRACT. Let \mathcal{A} be a unital C^* -algebra and let $\Phi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ be a unital n -positive linear map between C^* -algebras for some $n \geq 3$. We show that

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \Delta(A, \|\cdot\|) \Delta(B, \|\cdot\|)$$

for all operators $A, B \in \mathcal{A}$, where $\Delta(C, \|\cdot\|)$ denotes the operator norm distance of C from the scalar operators.

1. INTRODUCTION

Let $\mathbb{B}(\mathcal{H})$ stand for the algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, let $\|\cdot\|$ denote the operator norm and let I be the identity operator. For self-adjoint operators A, B the order relation $A \leq B$ means that $\langle A\xi, \xi \rangle \leq \langle B\xi, \xi \rangle$ ($\xi \in \mathcal{H}$). In particular, if $0 \leq A$, then A is called positive. If $\dim \mathcal{H} = k$, we identify $\mathbb{B}(\mathcal{H})$ with the algebra \mathcal{M}_k of all $k \times k$ matrices with entries in \mathbb{C} .

Let $\Delta(C, \|\cdot\|) = \inf_{\lambda \in \mathbb{C}} \|C - \lambda I\|$ be the $\|\cdot\|$ -distance of C from the scalar operators. It is known that $\Delta(C, \|\cdot\|) \leq \|C\|$ and $\Delta(C, \|\cdot\|) = c(C)$ for any normal operator C , where $c(C)$ denotes the radius of the smallest disk in the complex plane containing the spectrum $\sigma(C)$ of C ; see [11].

A linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ between C^* -algebras is said to be *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$. Every positive linear map Φ satisfies $\Phi(A^*) = \Phi(A)^*$ for all A . We say that Φ is unital if \mathcal{A}, \mathcal{B} are unital C^* -algebras and Φ preserves the identity. A linear map Φ is called *n -positive* if the map $\Phi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ defined by $\Phi_n([a_{ij}]_{n \times n}) = [\Phi(a_{ij})]_{n \times n}$ is positive, where $M_n(\mathcal{A})$ stands for the C^* -algebra of $n \times n$ matrices with entries in \mathcal{A} . It is known that $\|\Phi\| = 1$ for any unital n -positive linear map Φ . Φ is said to be *completely positive* if it is n -positive for every $n \in \mathbb{N}$. For a comprehensive account on completely positive maps see [9].

The Grüss inequality [6], as a complement of Chebyshev's inequality, states that if f and g are integrable real functions on $[a, b]$ and there exist real constants $\varphi, \phi, \gamma, \Gamma$ such that

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$\varphi \leq f(x) \leq \phi$ and $\gamma \leq g(x) \leq \Gamma$ hold for all $x \in [a, b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(\phi - \varphi)(\Gamma - \gamma).$$

This inequality has been investigated, applied and generalized by many mathematicians in different areas of mathematics; see [5] and references therein. Perić and Rajić proved a Grüss type inequality for unital completely bounded maps [10].

In what follows \mathcal{A} will stand for a unital C^* -algebra. In this paper we prove that if $\Phi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ is a unital n -positive linear map between C^* -algebras for some $n \geq 3$, then

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \Delta(A, \|\cdot\|) \Delta(B, \|\cdot\|) \quad (1.1)$$

for all operators $A, B \in \mathcal{A}$.

2. MAIN RESULT

To achieve our main result we need three lemmas. The first lemma can be deduced from [8, Theorem 2] by adding a necessary assumption $\Phi(A^*) = \Phi(A)^*$. We state it for the sake of convenience.

Lemma 2.1. *Let $\Phi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ be a unital $*$ - n -positive (not necessarily linear) map for some $n \geq 3$. Then*

$$\|\Phi(AB) - \Phi(A)\Phi(B)\|^2 \leq \|\Phi(|A^*|^2) - |\Phi(A^*)|^2\| \|\Phi(|B|^2) - |\Phi(B)|^2\| \quad (2.1)$$

for all operators $A, B \in \mathcal{A}$.

Proof. Let A and B be two operators in \mathcal{A} . We have

$$0 \leq \begin{bmatrix} A^* \\ B^* \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} A & B & I & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} A^*A & A^*B & A^* & 0 & \cdots & 0 \\ B^*A & B^*B & B^* & 0 & \cdots & 0 \\ A & B & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Due to Φ is n -positive, we get

$$0 \leq \begin{bmatrix} \Phi(A^*A) & \Phi(A^*B) & \Phi(A^*) & 0 & \cdots & 0 \\ \Phi(B^*A) & \Phi(B^*B) & \Phi(B^*) & 0 & \cdots & 0 \\ \Phi(A) & \Phi(B) & \Phi(I) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

whence

$$0 \leq \left[\begin{array}{cc|c} \Phi(A^*A) & \Phi(A^*B) & \Phi(A)^* \\ \Phi(B^*A) & \Phi(B^*B) & \Phi(B)^* \\ \hline \Phi(A) & \Phi(B) & I \end{array} \right]. \quad (2.2)$$

It is known that the matrix $\begin{bmatrix} R & T \\ T^* & S \end{bmatrix}$ is positive if and only if R, S are positive and $R \geq TS^{-1}T^*$, where S^{-1} denotes the (generalized) inverse of S . Using this fact and noting to (2.2) we get

$$\begin{bmatrix} \Phi(A^*A) & \Phi(A^*B) \\ \Phi(B^*A) & \Phi(B^*B) \end{bmatrix} \geq \begin{bmatrix} \Phi(A)^* \\ \Phi(B)^* \end{bmatrix} I^{-1} \begin{bmatrix} \Phi(A) & \Phi(B) \end{bmatrix} = \begin{bmatrix} \Phi(A)^*\Phi(A) & \Phi(A)^*\Phi(B) \\ \Phi(B)^*\Phi(A) & \Phi(B)^*\Phi(B) \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} \Phi(A^*A) - \Phi(A)^*\Phi(A) & \Phi(A^*B) - \Phi(A)^*\Phi(B) \\ \Phi(B^*A) - \Phi(B)^*\Phi(A) & \Phi(B^*B) - \Phi(B)^*\Phi(B) \end{bmatrix} \geq 0 \quad (2.3)$$

As noted in [2], the inequality (2.3) implies that

$$\|\Phi(A^*B) - \Phi(A)^*\Phi(B)\|^2 \leq \|\Phi(|A|^2) - |\Phi(A)|^2\| \|\Phi(|B|^2) - |\Phi(B)|^2\| \quad (2.4)$$

Replacing A by A^* in (2.4) we obtain (2.1). \square

Remark 2.2. The inequality (2.3) is known as the operator covariance-variance inequality; see [1] and references therein for more information.

The second lemma includes our main idea.

Lemma 2.3. *Let $\Phi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ be a unital positive linear map between C^* -algebras. Then*

$$\|\Phi(A^*A) - \Phi(A)^*\Phi(A)\| \leq \Delta(A, \|\cdot\|)^2 \quad (2.5)$$

for every normal operator $A \in \mathcal{A}$.

Proof. Let $A \in \mathcal{A}$ be a normal operator, and let \mathcal{C} denote the C^* -algebra generated by A, A^* and I . Then \mathcal{C} is commutative, so that the restriction of Φ to \mathcal{C} is completely positive by a known fact due to Choi; see [4, 9]. The Stinespring dilation theorem states that for any unital completely positive map $\Phi : \mathcal{C} \rightarrow \mathbb{B}(\mathcal{H})$ there exist a Hilbert space \mathcal{H} , an isometry

$V : \mathcal{H} \rightarrow \mathcal{K}$ and a unital $*$ -homomorphism $\pi : \mathcal{C} \rightarrow \mathbb{B}(\mathcal{K})$ such that $\Phi(A) = V^*\pi(A)V$. Now we have

$$\begin{aligned}
\|\Phi(A^*A) - \Phi(A)^*\Phi(A)\| &= \|\Phi((A - \lambda I)^*(A - \mu I)) - (\Phi(A - \lambda I))^*\Phi(A - \mu I)\| \\
&= \|V^*\pi((A - \lambda I)^*(A - \mu I))V - V^*(\pi(A - \lambda I))^*VV^*\pi(A - \mu I)V\| \\
&= \|V^*(\pi(A - \lambda I))^*(I - VV^*)\pi(A - \mu I)V\| \quad (\pi \text{ is } * \text{-homomorphism}) \\
&\leq \|\pi(A - \lambda I)\| \|\pi(A - \mu I)\| \quad (I - VV^* \text{ is a projection}) \\
&\leq \|A - \lambda I\| \|A - \mu I\| \quad (\pi \text{ is unital and norm decreasing})
\end{aligned}$$

for all $\lambda, \mu \in \mathbb{C}$. From this it follows that

$$\|\Phi(A^*A) - \Phi(A)^*\Phi(A)\| \leq \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\| \inf_{\mu \in \mathbb{C}} \|A - \mu I\| = \Delta(A, \|\cdot\|)^2.$$

□

Remark 2.4. Passing the proof of previous lemma, one can easily deduce that in the case of a unital completely positive map $\Phi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ the statement of Lemma 2.3 is valid for every $A \in \mathcal{A}$.

The next lemma is well known.

Lemma 2.5. [7, Theorem 1] *Let $A \in \mathcal{A}$ and $\|A\| < 1 - (2/m)$ for some integer m greater than 2. Then there are m unitary elements $U_1, \dots, U_m \in \mathcal{A}$ such that $mA = U_1 + \dots + U_m$.*

We are ready to present our main result.

Theorem 2.6. *Let $\Phi : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H})$ be a unital n -positive linear map between C^* -algebras for some $n \geq 3$. Then*

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \Delta(A, \|\cdot\|) \Delta(B, \|\cdot\|) \quad (2.6)$$

for all operators $A, B \in \mathcal{A}$.

Proof. Using Lemmas 2.1 and 2.3 we obtain (2.6) for normal operators $A, B \in \mathcal{A}$.

Let $0 \neq A \in \mathcal{A}$. Let $m > 2$ be an integer and $M = \frac{m^2+2}{m^2-2m}\|A\|$. Then

$$\|A/M\| = \frac{m^2 - 2m}{m^2 + 2} < 1 - \frac{2}{m}.$$

By Lemma 2.5, there are unitaries $U_1, \dots, U_m \in \mathcal{A}$ such that $A = \frac{M}{m} \sum_{j=1}^m U_j$. Hence for any normal operator $B \in \mathcal{A}$ we have

$$\begin{aligned}
\|\Phi(AB) - \Phi(A)\Phi(B)\| &= \frac{M}{m} \left\| \Phi \left(\sum_{j=1}^m U_j B \right) - \Phi \left(\sum_{j=1}^m U_j \right) \Phi(B) \right\| \\
&\leq \frac{m^2 + 2}{m^3 - 2m^2} \|A\| \sum_{j=1}^m \|\Phi(U_j B) - \Phi(U_j)\Phi(B)\| \\
&\leq \frac{m^2 + 2}{m^3 - 2m^2} \|A\| \sum_{j=1}^m \|U_j\| \|B\| \quad (\text{by (2.6) for normal operators}) \\
&\leq \frac{m^2 + 2}{m^2 - 2m} \|A\| \|B\|.
\end{aligned}$$

Letting $m \rightarrow \infty$, we infer that

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \|A\| \|B\| \quad (2.7)$$

for arbitrary A and normal B . By repeating the same argument for arbitrary B and by using (2.7), we deduce that

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \|A\| \|B\|$$

for all $A, B \in \mathcal{A}$.

Next we observe that

$$\begin{aligned}
\|\Phi(AB) - \Phi(A)\Phi(B)\| &= \|\Phi((A - \lambda I)(B - \mu I)) - \Phi(A - \lambda I)\Phi(B - \mu I)\| \\
&\leq \|A - \lambda I\| \|B - \mu I\|
\end{aligned}$$

for all $\lambda, \mu \in \mathbb{C}$. Thus,

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\| \inf_{\mu \in \mathbb{C}} \|B - \mu I\| = \Delta(A, \|\cdot\|) \Delta(B, \|\cdot\|)$$

for all $A, B \in \mathcal{A}$. This proves the theorem. \square

Remark 2.7. Let us remark here that the inequality (2.6) does not have to hold if Φ is assumed only to be a unital positive linear map. To see this, let us choose $\Phi : \mathcal{M}_2 \rightarrow \mathcal{M}_2$ to be the transpose map. It is known (see e.g. [9]) that such a map is positive, but not 2-positive. Hence, Φ is not a 3-positive map. Then, for matrices

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

we have

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| = 6.$$

Since A and B are positive matrices such that $\sigma(A) = \{0, 5\}$ and $\sigma(B) = \{1, 4\}$, we conclude that $\Delta(A, \|\cdot\|) = c(A) = \frac{5}{2}$ and $\Delta(B, \|\cdot\|) = c(B) = \frac{3}{2}$. Therefore,

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| = 6 > \frac{15}{4} = \Delta(A, \|\cdot\|) \Delta(B, \|\cdot\|).$$

We do not know whether (2.6) holds if a map Φ is assumed to be 2-positive.

Theorem 2.6 extends the result obtained in [10, Corollary 1] to the case of n -positive linear maps, where $n \geq 3$. Some applications of Theorem 2.6 on completely positive maps were also given in [10]. The first example of n -positive map ($n \geq 2$), which is not completely positive, was obtained by Choi in [3]. He showed that the map $\Phi : \mathcal{M}_k \rightarrow \mathcal{M}_k$ defined by

$$\Phi(T) = (k-1)\text{tr}(T)I - T \quad (T \in \mathcal{M}_k)$$

is $(k-1)$ -positive, but not k -positive. (Here 'tr' denotes the trace.) Later, Takasaki and Tomiyama [12] introduced a way to construct new examples of $(k-1)$ -positive linear maps from \mathcal{M}_k to \mathcal{M}_k which are not k -positive.

In the next corollary we apply Theorem 2.6 on Choi's $(k-1)$ -positive linear map.

Corollary 2.8. *Let $A, B \in \mathcal{M}_k$, where $k \geq 4$. Then*

$$\begin{aligned} & \|(k^2 - k - 1)\text{tr}(AB)I - kAB - (k-1)\text{tr}(A)\text{tr}(B)I + \text{tr}(B)A + \text{tr}(A)B\| \\ & \leq \frac{(k^2 - k - 1)^2}{k-1} \Delta(A, \|\cdot\|) \Delta(B, \|\cdot\|). \end{aligned}$$

Proof. Define

$$\Phi(T) = \frac{k-1}{k^2 - k - 1} \text{tr}(T)I - \frac{1}{k^2 - k - 1} T \quad (T \in \mathcal{M}_k).$$

By Theorem 2.6 we have

$$\|\Phi(AB) - \Phi(A)\Phi(B)\| \leq \Delta(A, \|\cdot\|) \Delta(B, \|\cdot\|)$$

since Φ is a unital $(k-1)$ -positive linear map. An easy computation shows that

$$\begin{aligned} & \Phi(AB) - \Phi(A)\Phi(B) \\ & = \frac{k-1}{(k^2 - k - 1)^2} [(k^2 - k - 1)\text{tr}(AB)I - kAB - (k-1)\text{tr}(A)\text{tr}(B)I + \text{tr}(B)A + \text{tr}(A)B], \end{aligned}$$

from which the result immediately follows. \square

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¹ DEPARTMENT OF PURE MATHEMATICS, CENTER OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS), FERDOWSI UNIVERSITY OF MASHHAD, P.O. BOX 1159, MASHHAD 91775, IRAN.
E-mail address: moslehian@ferdowsi.um.ac.ir and moslehian@ams.org

² FACULTY OF MINING, GEOLOGY AND PETROLEUM ENGINEERING, UNIVERSITY OF ZAGREB, PIEROT-TIJEVA 6, 10000 ZAGREB, CROATIA
E-mail address: rajna.rajic@zg.t-com.hr